

## Exercise: concepts from chapter 5

Reading: Fundamentals of Structural Geology, Ch 5

1) Study the oöids depicted in Figure 1a and 1b.



Figure 1a

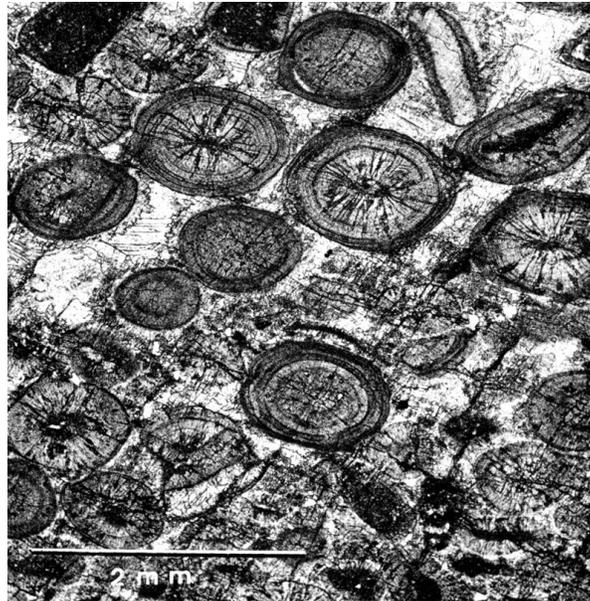


Figure 1b

Figure 1. Nearly undeformed (1a) and significantly deformed (1b) oöids with spherulitic cores viewed in thin section. Matrix is calcite with some mud. Samples are from near Harrisonburg, VA. Reprinted from Cloos (1971, Plates 9 and 11) with permission of The John Hopkins University Press.

- a) Assume that the thin sections of Figure 1 lie in a principal plane of the deformation. Measure and record the lengths and orientations of the principal axes of several oöids from each thin section.
  - b) Estimate the un-deformed radius of each oöid assuming no volume change during the deformation. Calculate and record the stretch in the two principal directions for each oöid.
  - c) Compare the values of the principal stretches for the different oöids. How uniform is the deformation within these thin sections? Point out examples of non-uniform deformation within particular oöids?
  - d) Cloos suggested that some oöids have "growth aprons" developed after the deformation. How would you compensate for these aprons when estimating the stretch?
- 2) One way to build intuition about deformation is to witness the deformation of familiar objects. Here the objects are brachiopods, which come in many different shapes (Fig. 2).

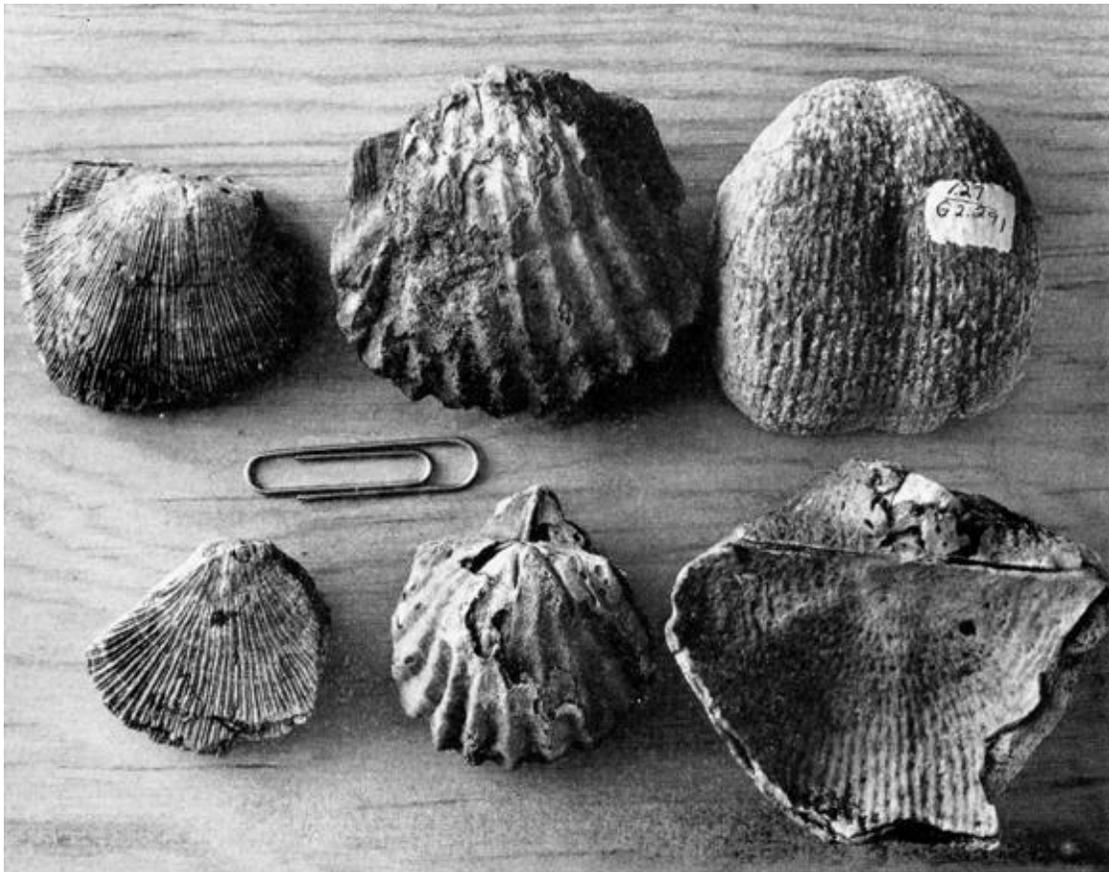


Figure 2. Fossil brachiopods from the Kaibab limestone, Grand Canyon, Arizona. Specimens at the Museum of Northern Arizona, Flagstaff. Photograph from Shelton, 1966, p. 285.

Most brachiopods contain two recognizable lines (Fig. 3): the hinge line and the line of symmetry or central plication. These lines are perpendicular to each other in the undeformed state.

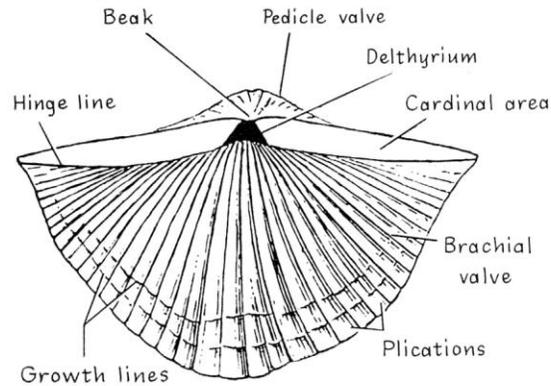


Figure 3. External features of a brachiopod shell. After H. W. Shimer, 1914. From Woodford, 1965, p. 480.

To model the deformation of brachiopods we idealize their shapes as shown in Figure 4.

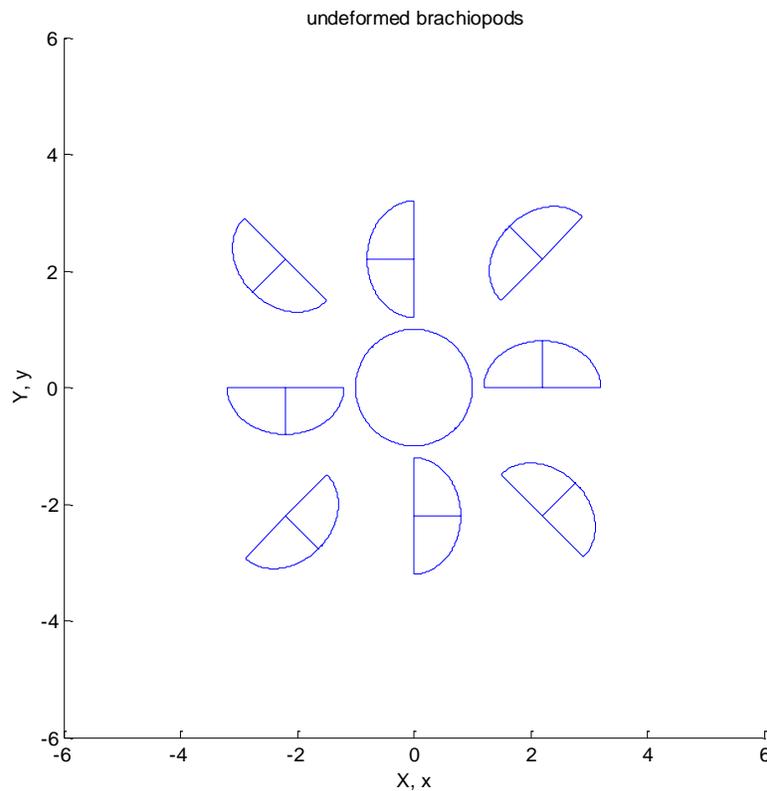


Figure 4. Idealized brachiopods and a circle lying in a bedding surface that is the  $(X, Y)$ -plane. This configuration of lines and curves is used for the deformation analysis.

This exercise provides a visualization of deformation that emphasizes the change in orientation of orthogonal material lines: in this case the hinge line and central plication of the brachiopod. In other words this exercise will help you build your intuition about *shearing* deformation.

We consider a variety of deformations, some of which you might not expect to be associated with shearing, but all of which are two-dimensional, homogeneous, and can be described by the following *linear transformations*:

$$\begin{aligned} x &= F_{xx}X + F_{xy}Y \\ y &= F_{yx}X + F_{yy}Y \end{aligned} \quad (1)$$

Here  $(x, y)$  are the final coordinates of a particle;  $(X, Y)$  are the initial coordinates of that particle; and the coefficients  $(F_{xx}, F_{xy}, F_{yx}, F_{yy})$  are constants that determine the style of deformation. An example is given in Figure 5 where:

$$F_{xx} = 3/2, F_{xy} = 1/2, F_{yx} = 1/2, F_{yy} = 2/3 \quad (2)$$

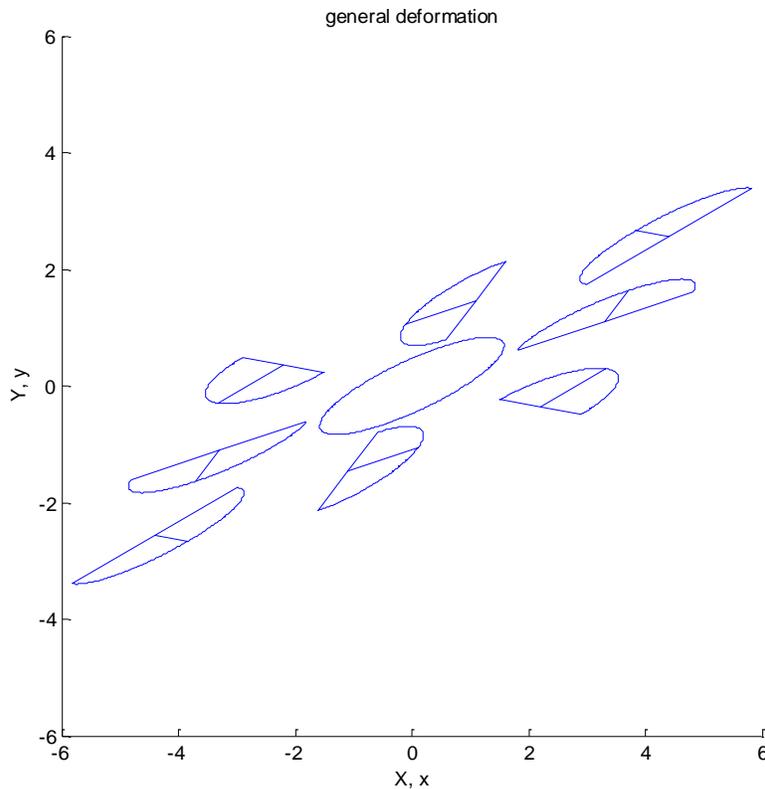


Figure 5. Deformed idealized brachiopods and strain ellipse for the deformation described by the constants in (2) and the initial state depicted in Figure 4.

- a) Construct a set of brachiopods similar to those in Figure 4, each with a straight hinge line, a perpendicular line of symmetry, and a rounded valve perimeter.

Avoid overlapping of the brachiopods. Place a circle in the center of the field of brachiopods to track the orientations and magnitudes of the principal stretches as associated with a strain ellipse.

b) *Uniaxial extension* is defined as:

$$F_{xx} \neq 1, \quad F_{xy} = 0, \quad F_{yx} = 0, \quad F_{yy} = 1 \quad (3)$$

The bedding surface is stretched or shortened along the  $x$ -axis only. Does this deformation result in shearing? If your answer is yes, describe which orientations of originally orthogonal material lines are sheared and which are not. What are the orientations and magnitudes of the principal stretches?

c) A general *biaxial extension* is defined:

$$F_{xx} \neq 1, \quad F_{xy} = 0, \quad F_{yx} = 0, \quad F_{yy} \neq 1 \quad (4)$$

The bedding surface is stretched or shortened along both the  $x$ - and  $y$ -axis. Consider two special cases, so called *pure shear*:

$$F_{xx} = 1 / F_{yy} \quad (5)$$

and *pure dilation*:

$$F_{xx} = F_{yy} \quad (6)$$

Compare and contrast the two special cases in terms of the question: does this deformation result in shearing? If your answer is yes, describe which orientations of originally orthogonal material lines are sheared and which are not. Also compare and contrast the orientations and magnitudes of the principal stretches.

d) *Simple shearing* is defined:

$$F_{xx} = 1, \quad F_{xy} = \tan \Psi, \quad F_{yx} = 0, \quad F_{yy} = 1 \quad (7)$$

Here  $\Psi$  is the angle of shearing: the change in orientation of the material line originally coincident with the  $Y$ -axis. For this exercise choose  $\Psi = 45^\circ$  so  $F_{xy} = 1$ .

Does this material line elongate or shorten? What is the change in orientation of the material line originally coincident with the  $X$ -axis? Does this material line elongate or shorten? Do any material lines shorten during simple shearing? If so, how are they oriented in the initial state with respect to the shear planes? What are the orientations and magnitudes of the principal stretches? What is simple about simple shearing?

3) One of the first and most important tasks to perform when embarking on a study of deformation is to determine whether one may take advantage of the powerful tools of continuum mechanics available in the realm of small strains, for example as used in elasticity theory. To make this assessment one needs to understand how the general description for an arbitrary deformation is specialized to infinitesimal strain and rotation.

- a) Consider the following description of deformation for the material line of particles along the infinitesimal vector  $d\mathbf{X}$  in the initial state that is translated, rotated, and stretched to lie along the infinitesimal vector  $d\mathbf{x}$  in the final state:

$$\begin{aligned} dx &= F_{xx}dX + F_{xy}dY \\ dy &= F_{yx}dX + F_{yy}dY \end{aligned} \quad (8)$$

Draw a figure that illustrates and labels the position vectors  $\mathbf{X}$  and  $\mathbf{x}$  for the particle in the initial state at the tail of  $d\mathbf{X}$  and in the final state at the tail of  $d\mathbf{x}$ ; the displacement vector  $\mathbf{u}$  for that particle; the vectors  $d\mathbf{X}$  and  $d\mathbf{x}$  representing the material line in these two states; the components of these vectors ( $dX, dY$ ) and ( $dx, dy$ ); the lengths (magnitudes) of the material lines,  $dS$  and  $ds$ ; and the orientations,  $\Theta$  and  $\theta$ , of the material lines relative to the  $(X, x)$  axis.

- b) Using the following trigonometric relations, which should be consistent with your illustration, derive the equation for the square of the stretch of the material line, that is  $(ds)^2/(dS)^2$ , in terms of the components of the deformation gradient tensor,  $F_{ij}$ , and the orientation of the material line in the initial state,  $\Theta$ . Use double angle formulae for the trigonometric relations.

$$\begin{aligned} dX &= dS \cos \Theta, & dY &= dS \sin \Theta \\ dx &= ds \cos \theta, & dy &= ds \sin \theta \end{aligned} \quad (9)$$

- c) The normal (longitudinal) strain in the arbitrary direction of the material line is defined:

$$\varepsilon_n = \frac{ds - dS}{dS} \quad (10)$$

Use this definition to write an approximation for the square of the stretch that omits second order terms. Under these conditions show that the normal strain and the stretch are related as:

$$\varepsilon_n \approx (S^2 - 1)/2 \quad (11)$$

- d) All of the components of the deformation gradient tensor,  $F_{ij}$ , enter the equation for the square of the stretch as squares or products. Therefore omitting terms of this order would eliminate all terms. Rewrite the equation for the square of the stretch

in terms of the displacement gradients, omitting squares or products of the displacement gradients to find an approximation for small normal strains.

- e) Combine your results from questions 3c) and 3d) to write an equation for the normal strain in terms of the displacement gradients for the conditions of small displacement gradients and small normal strains. Identify the components of the infinitesimal strain in terms of the displacement gradients and write the normal strain in terms of the infinitesimal strain components.
- f) Suppose  $F_{xx}$  is the only non-zero component of the displacement gradient tensor and consider a material line initially oriented such that  $\Theta = 0$ . Identify which displacement gradient plays a role in the small strain approximation for  $F_{xx}$  and determine limits for this gradient that would assure an error of less than 10% in the computation of the square of the stretch.

4) In this exercise we consider the small strains and rotations near a model fault using a solution from elasticity theory. In later chapters we describe the elastic properties of rock and derive the governing equations for linear elasticity. Here we focus only on the kinematics. The model fault (Fig. 6) slips in a right-lateral sense driven by the stress drop  $\Delta\sigma = \sigma_{yx}^r - \sigma_{yx}^f$ , and resisted by the elastic stiffness of the surrounding brittle solid,  $G/(1-\nu)$ , where  $G$  is the shear modulus (typically  $\sim 1 - 100$  GPa for rock) and  $\nu$  is Poisson's ratio (typically  $\sim 0.1 - 0.3$  for rock).

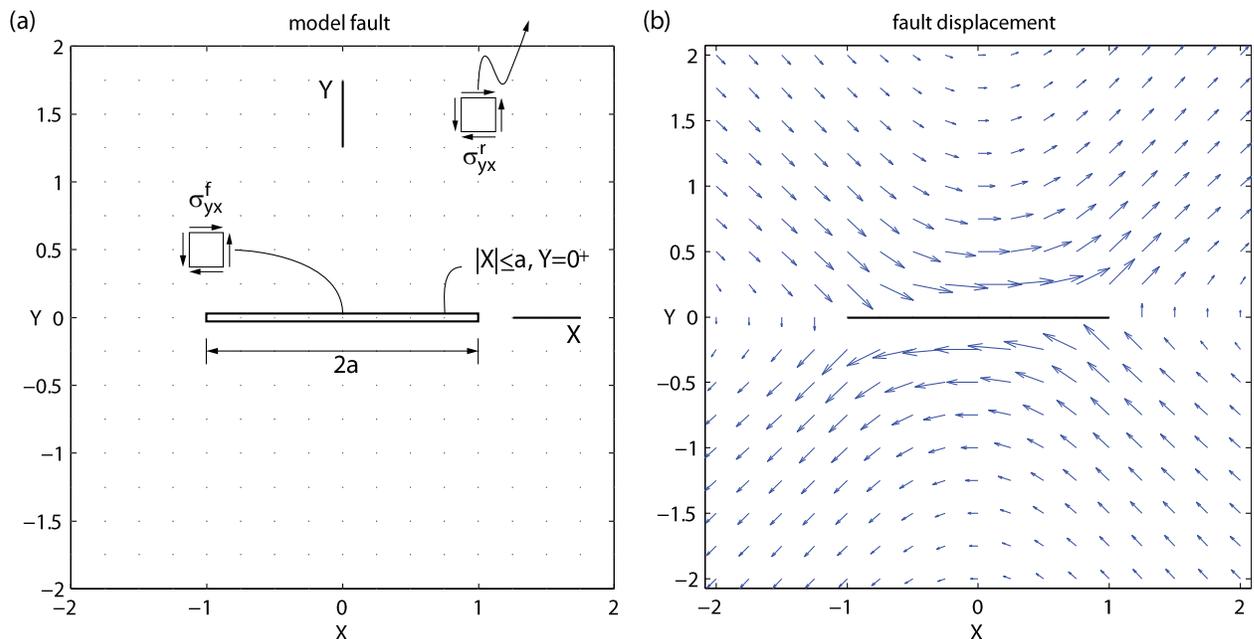


Figure 6. (a) Model fault in an elastic body subject to remote shear stress. (b) Displacement field due to slip on model fault.

The displacement components along the surface of the fault,  $|x| \leq a, y = 0^+$ , are (Pollard and Segall, 1987):

$$u_x = \Delta\sigma \left( \frac{1-\nu}{G} \right) (a^2 - x^2)^{1/2}, \quad u_y = \Delta\sigma \left( \frac{1-2\nu}{2G} \right) x \quad (12)$$

The displacement components along the positive  $y$ -axis, that is on  $x=0, y \geq 0$ , are (Pollard and Segall, 1987):

$$u_x = \Delta\sigma \left( \frac{1-\nu}{G} \right) \left[ (a^2 + y^2)^{1/2} - y \right] + \Delta\sigma \left( \frac{1}{2G} \right) \left[ y^2 (a^2 + y^2)^{-1/2} - y \right], \quad u_y = 0 \quad (13)$$

- Evaluate the three small strain components at the middle of the model fault where  $x=0$  and  $y=0^+$ . Describe each component with reference to material lines at that point and oriented in the two coordinate directions. Explain the magnitude and sign of each component relating these to slip on the model fault and the elastic stiffness of the surrounding rock.
- Evaluate the small rotation at the middle of the model fault where  $x=0$  and  $y=0^+$ . Describe the rotation with reference to material lines at that point and oriented in the two coordinate directions. Explain the magnitude and sign of the rotation relating this to slip on the model fault and the elastic stiffness of the surrounding rock.
- Use your result from a) and b) to write down the elements of the matrix equation for the partial derivatives of displacement in terms of a sum of a shear strain matrix and a rotation matrix. For simplicity let Poisson's ratio be zero for this exercise.

$$\begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} = \quad (14)$$

- According to the second of (12) the displacement component  $u_y$  acting perpendicular to the fault surfaces is linearly distributed along the fault. This indicates that the fault surfaces remain planar as they rotate. Use the second of (12) with Poisson's ratio set to zero to find  $u_y$  at the fault tip. Compare this to the lower left element of (14) which accounts for the combined strain and rotation at the fault middle. Are they consistent?